Binomial Line Cox Processes

This talk: Part 1 - Characterization and Meta-Distributions

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Part 2 - *Applications in Wireless Networks and Automotive Radars* is a separate talk.

Acknowledgements

- Joint work with Md. Taha Shah (Ph.D. student at IIT Delhi) and Martin Haenggi (EE, University of Notre Dame).
- Thanks to Maruti-Suzuki India Limited for supporting a part of this work.

- The talk (Part 1) is derived from the following three papers:
  - G. Ghatak, *Binomial Line Processes: Distance Distributions*, IEEE Transactions on Vehicular Technology, doi: 10.1109/TVT.2021.3134834.
  - M. T. Shah, *et al. Analyzing Wireless Networks using Binomial Line Cox Processes* in the IEEE WiOpt 2023 Workshop on Spatial Stochastic Models for Wireless Networks - SpaSWiN.
  - Md. T. Shah, *et al. Binomial Line Cox Processes: Statistical Characterization and Applications in Wireless Network Analysis*, IEEE Transactions on Wireless Communication (accepted, May 2024).

Overview of Today's Talk

Context

Modern Networks are Increasingly AdHoc

(a) (b)

Figure: BS discovery at (a) IISc Campus and (b) ISI Bangalore Campus. Source: OpenCellID.

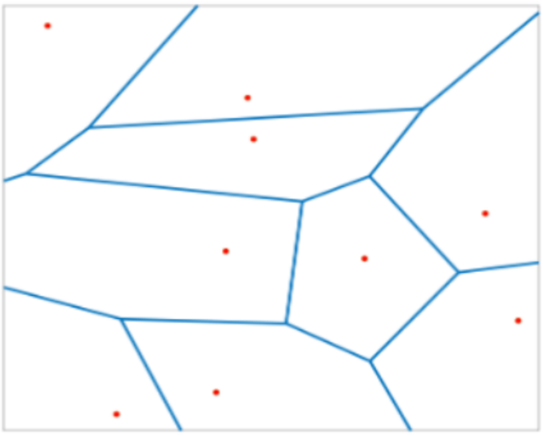
Modern Networks are Increasingly AdHoc

(a) (b)

Figure: BS discovery at (a) IISc Campus and (b) ISI Bangalore Campus. Source: OpenCellID.

- Are these two scenarios to be studied separately?

Solution - Stochastic Networks and Point Processes



Next-Generation Wireless Deployment

(a)

(b)

Figure: (a) Example first-generation mm-Wave deployment. (b) Verizon's small cell deployment near stadiums.

Other examples

- New York City has already deployed such small cells focusing on high-speed connectivity for pedestrian users.
- South Korea: SK Telecom, 28GHz, outdoor pedestrian.
- Taiwan: Nokia and CHT.
- Verizon deployed 5G (with mm-wave) on street lights in Sacramento.
- AT&T deployed 5G (with mm-wave) on smart lamp-posts in San Jose.

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Ok, so deployment along streets. What else?

Coverage Footprint from a 5G Lamppost

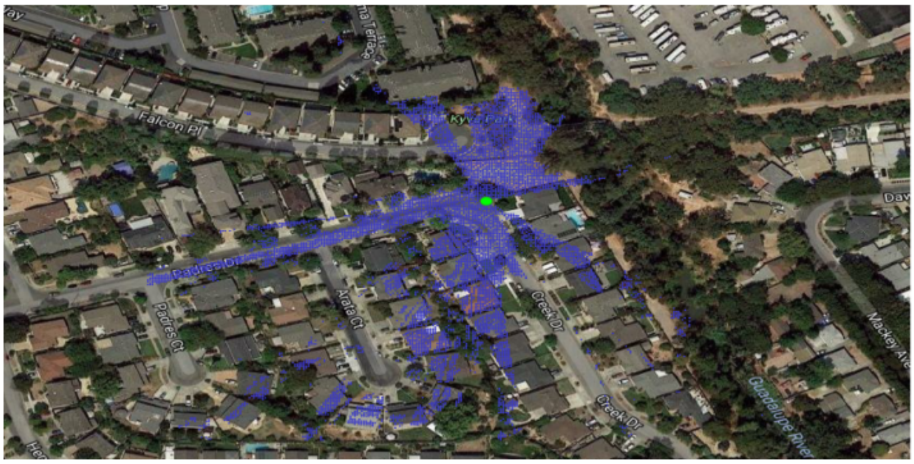

Figure: Coverage of a lamp-post mm-wave small cell in San Jose.

Motivation and History of Line Processes

Networks on Lines - Other Examples

- Movement of vehicular nodes.

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Let's look at some history...

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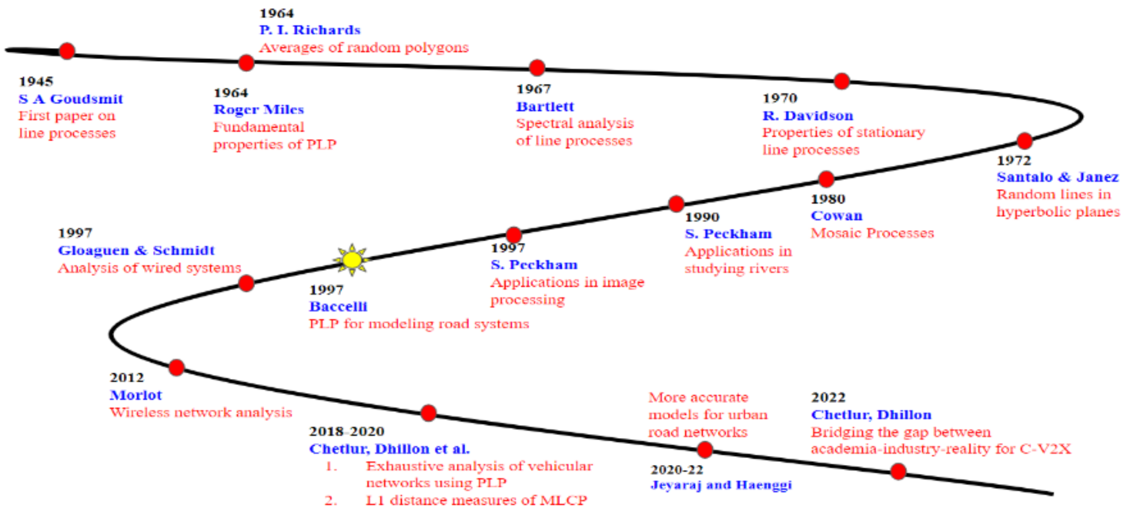
History of Line Process Research



[A part of the illustration is from Prof. Dhillon's book/talk.]

Candidate Models for Emulating Streets

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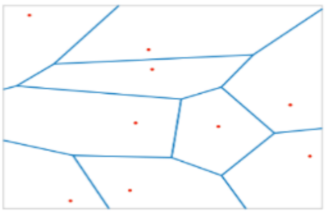
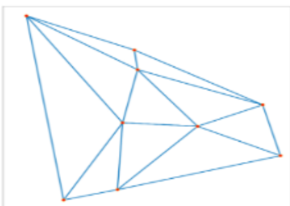
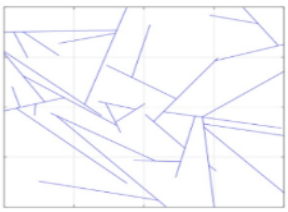
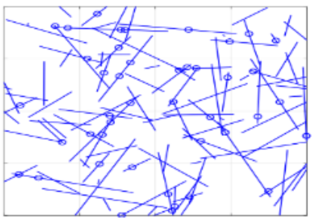
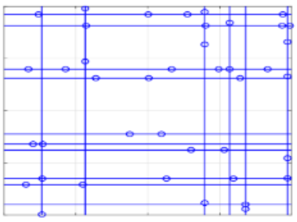
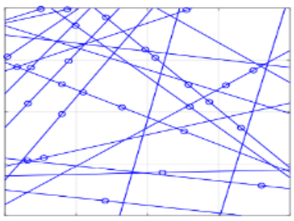

Poisson Voronoi Tessellation


Poisson Delaunay Triangulation


Poisson Lilypond Model


Poisson Stick Process


Manhattan Line Process


Poisson Line Process

Some Observations

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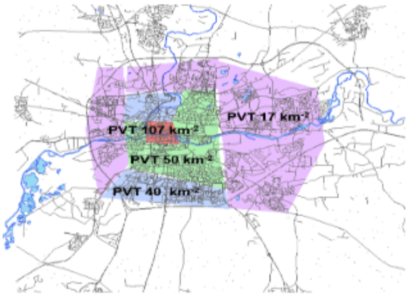
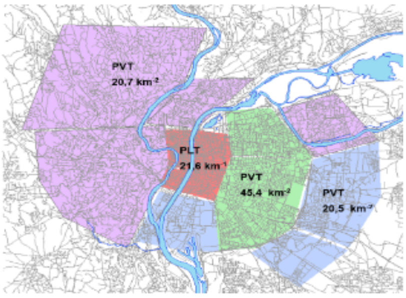
- Homogeneity → characterization is motion-invariant - one can study the network from the perspective of a single point, e.g., at the origin.
- From the perspective of a single city, the street networks are inhomogeneous - denser streets in the city center, not so much in suburbs.
- Solutions need to be adaptive -
  - MAC protocols should adapt based on the location.
  - Deployment planning must take into account the location - e.g., where is the nearest eV charging point? nearest bus stop?
  - V2X load is directly dependent on the length of streets in a coverage area; load-balancing and cell-breathing need to be adaptive.
  - ...

Emulating Non-Homogeneous Streets



(a)



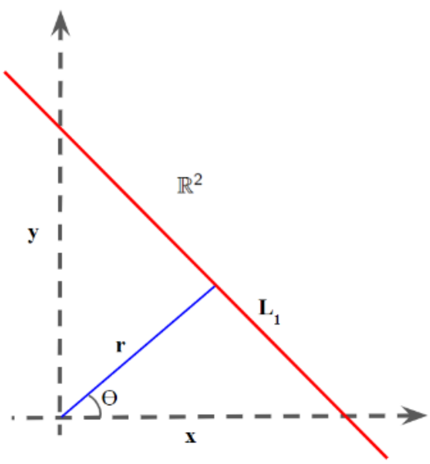
(b)

Figure: (a) PVT and (b) PVT + PLT models for fitting for streets of Lyon [2]. For a PVT, the model parameter is the density of the underlying Poisson point process (PPP).

[2] C. Gloaguen, "Modelisation macroscopique geometrique des reseaux d'acces en telecommunication," Presentation at Soci´et´e de Math´ematiques Appliqu´ees et Industrielles. Link: http://smai.emath.fr/spip/IMG/ppt, 2010

Binomial Line Processes

Line Process - Construction



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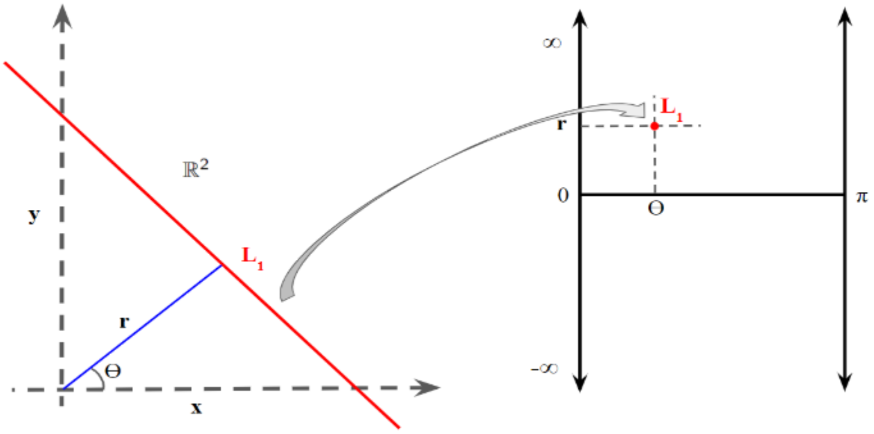

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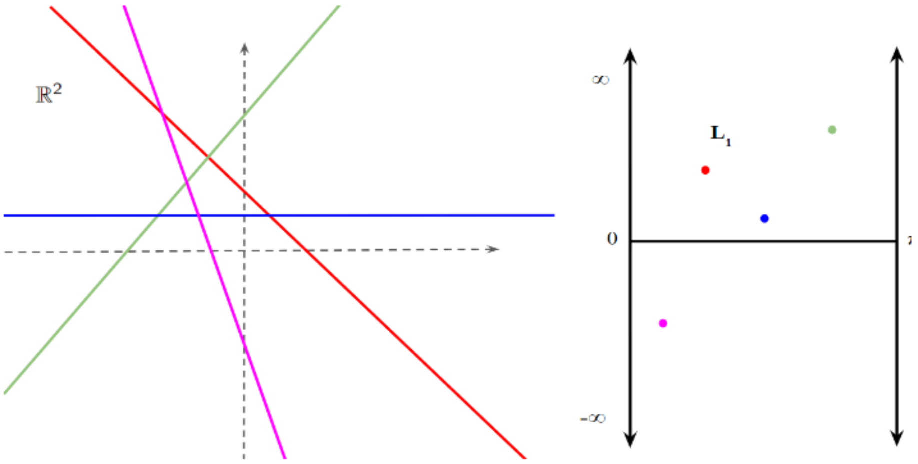

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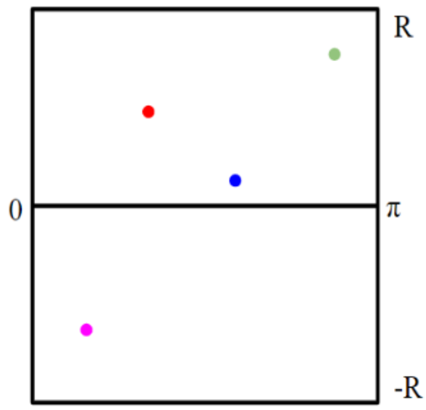

- **Key:** Fixed number of points (n_B), uniformly distributed in a closed compact set $\mathcal{D} \triangleq [-R, R] \times [0, \pi)$.

BLP - Definition

Binomial Line Process (BLP)

A binomial line process (BLP) $\mathcal{L}$ is a collection of n_B lines in the two-dimensional Euclidean plane. Formally,

$$\mathcal{L} \subset Q \triangleq \bigcup_{r \in [-R,R], \theta \in [0,\pi)} \{(x,y) \in \mathbb{R}^2 \colon x\cos\theta + y\sin\theta = r\}. \quad (1)$$

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- A point $(\theta_i, r_i) \in \mathcal{D}$, corresponding to a line $L_i \in \mathcal{L}$, the generating point of L_i.
- Generating points form a BPP in $\mathcal{D}$.

Instance Visualization

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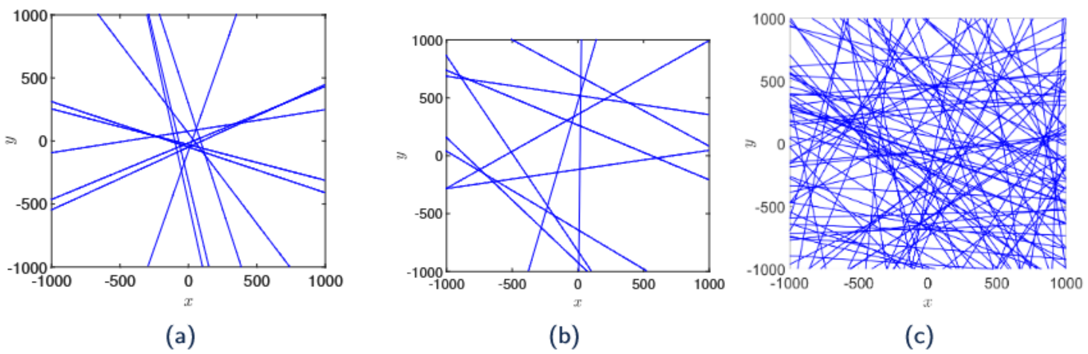

Figure: (a) A BLP with $n_B = 10$ and $R = 100$, (b) A realization of a PLP conditioned on 10 lines generated in the window of interest, and (c) A PLP whose underlying PPP has intensity $\frac{n_B}{2\pi R}$ with $n_B = 10$ and $R = 100$. Note: Here R is the radius of the circle in which BLP lines are generated.

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- However, isotropic $\to$ the properties as seen from a point depends only on its distance from the origin and not its orientation.
- Accordingly, without loss of generality, let us consider a test point located at $(0, r_0)$.

Distance to the Nearest BLP Line

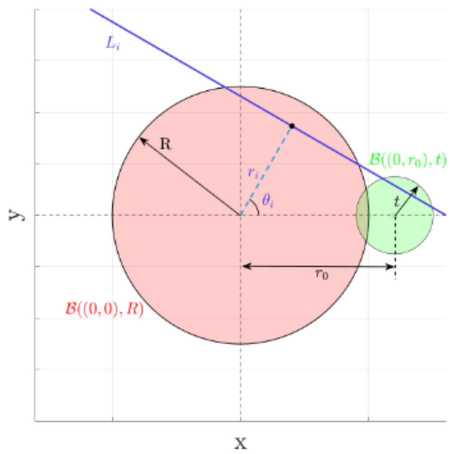
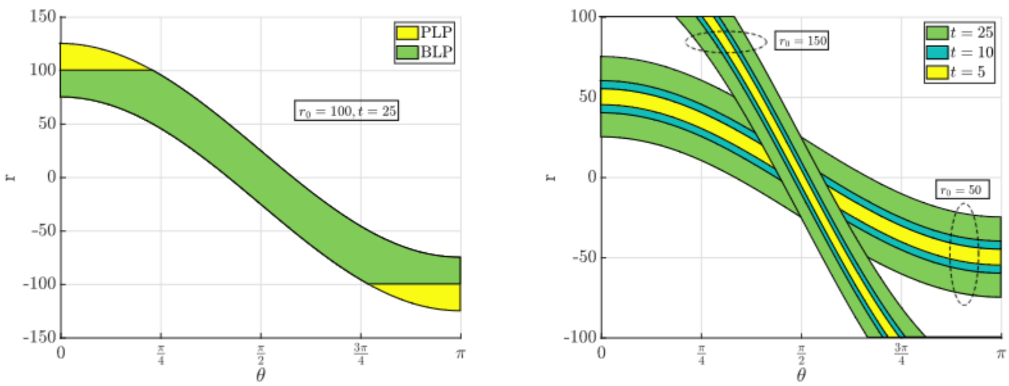


Figure: Illustration of the construction of a BLP and intersecting lines on $\mathcal{B}((0, r_0), t)$.

Domain Bands



(a) (b)

Figure: (a) Illustration of the domain bands for a PLP and a BLP with $R = 100$. (b) Domain bands for different values of t and r_0. Here $R = 100$. Note that when $r_0 + t \leq R$, the domain bands for PLP and BLP coincide.

Area of the Domain Band

Theorem

The area of the domain band $\mathcal{D}_{\mathrm{B}}(r_0, t)$ for a BLP is

$$A_{\mathrm{D}}(r_0,t) = \begin{cases} 2\pi t; & \text{for } r_0 + t \le R \\ 2\pi t - 2r_0\sqrt{1-\left(\frac{R-t}{r_0}\right)^2} + 2\,(R-t)\cos^{-1}\left(\frac{R-t}{r_0}\right); & \text{for } r_0 + t > R \text{ and } r_0 - t \le R \\ 2\pi t - 2r_0\left(\sqrt{1-\left(\frac{R-t}{r_0}\right)^2} - \sqrt{1-\left(\frac{R+t}{r_0}\right)^2}\right) + 2\,(R-t)\cos^{-1}\left(\frac{R-t}{r_0}\right) & \\ \qquad -2\,(R+t)\cos^{-1}\left(\frac{R+t}{r_0}\right); & \text{for } r_0 - t \ge R. \end{cases} \quad (2)$$

Distance to the nearest street?

Corollary

[Void Probability] The probability that no line of the BLP intersects with $\mathcal{B}((0,r_0),t)$ is

$$\mathcal{V}_{\mathrm{BLP}}(n_{\mathrm{B}}, \mathcal{B}((0,r_0),t)) = \left(1 - \frac{A_{\mathrm{D}}(r_0,t)}{2\pi R}\right)^{n_{\mathrm{B}}},$$

where, n_{B} is the number of lines of the BLP $\mathcal{L}$ and A_{D} is area of the domain bands.

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Corollary

The CDF of the distance to the nearest line of the BLP from a test point at $(0, r_0)$ is,

$$F_d(t) = 1 - \mathcal{V}_{\mathrm{BLP}}(n_{\mathrm{B}}, \mathcal{B}((0,r_0),t)).$$

Line Length Density

Line Length Density

Definition

The line length measure is

$$\mathcal{R}(S) = n_{\mathrm{B}}\, \mathbb{E}\left(|L \cap S|_1\right), \quad S \subset \mathbb{R}^2,$$

where $|\cdot|_1$ is the Lebesgue measure in one dimension and L is a line of the BLP. The corresponding radial density is

$$\rho(r) = \lim_{u \to 0} \frac{\mathcal{R}\big(\mathcal{B}((0,0), r+u) \setminus \mathcal{B}((0,0), r)\big)}{\pi\left(2u + u^2\right)}.$$

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The line length measure follows by integrating $\rho(r)$, i.e.,

$$\mathcal{R}(S) = \int_S \rho(|\mathbf{x}|)\, \mathrm{d}\mathbf{x}, \quad S \subset \mathbb{R}^2.$$

Line Length Density

Theorem

For a BLP generated by n_{B} lines within a disk of radius R,

$$\rho(r)=\begin{cases}\frac{n_{\mathrm{B}}}{2R}, & \text{if } r \leq R\\ \frac{n_{\mathrm{B}}}{\pi R}\arcsin\left(\frac{R}{r}\right) & \text{if } r > R.\end{cases}$$

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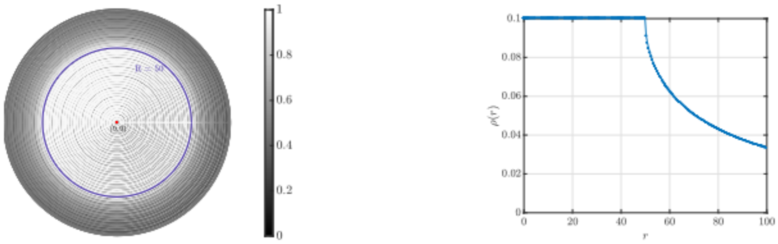

(a) (b)

Figure: (a) Ratio of the line length measure to the area in concentric annuli of equal width $w = 2$. Here, $R = 50$. (b) Line length density $\rho(r)$ for $R = 50$ and $n_{\mathrm{B}} = 10$.

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Theorem

The radial intersection density at a distance r from the origin for a BLP generated by n_{B} lines within a disk of radius R is

$$\rho_{\times}(r) = \begin{cases} \frac{n_{\mathrm{B}}(n_{\mathrm{B}}-1)}{4\pi R^2}, & \text{if } r \leq R, \\ \frac{n_{\mathrm{B}}(n_{\mathrm{B}}-1)}{4\pi^2 R^2 r}\left(2r \arcsin\left(\frac{R}{r}\right) - \frac{2R}{r}\sqrt{r^2 - R^2}\right) & \text{if } r > R. \end{cases}$$

Radial Intersection Density: Proof

- Let $S = \mathcal{B}((0,0), t)$, and consider L_0 generated at $(0, r_0)$ where $0 \leq r_0 \leq \min\{t, R\}$.
- First, we determine the domain band $\mathcal{D}_{\times}$ corresponding to the intersection on L_0
- For a θ, the range of r that correspond to the domain band is,
  $\max\left\{-R, \left(r_0 \cos\theta - \sqrt{t^2 - r_0^2}\sin\theta\right)\right\} \leq r_i \leq \min\left\{R, \left(r_0 \cos\theta + \sqrt{t^2 - r_0^2}\sin\theta\right)\right\}.$
- For $t > R$, the domain band gets clipped to R (upper) and $-R$ (lower) limits

$$A_{D_{\times}}(t) = \begin{cases} \pi t, & \text{if } t \leq R, \\ \frac{2}{R}\left(t^2 \arcsin\left(\frac{R}{t}\right) + 2R^2 \arccos\left(\frac{R}{t}\right) - \sqrt{t^2 - R^2}\right) & \text{if } t > R. \end{cases}$$

Radial Intersection Density: Proof

- Accordingly, the probability that a line intersects L_0 within S is obtained as $\mathcal{P}_\times(t) = \frac{A_{D_\times}(t)}{2\pi \min\{t,R\}}$.
- Now, let us assume that k lines are generated in S. Each of these intersects L_0 with probability $\mathcal{P}_\times(t)$. As a result, the average number of intersections on L_0 within S from the k lines is

$$\mathcal{N}' = \sum_{j=0}^{k} j \binom{k}{j} \left(\mathcal{P}_\times(t \,|\, t \leq R)\right)^j \left(1 - \mathcal{P}_\times(t \,|\, t \leq R\right)^{k-j} = \frac{k}{2}.$$

Radial Intersection Density: Proof

Finally, in order to determine the average number of intersections on all the lines within S, we take the expectation over the number of lines generated within S. This is evaluated as

$$\mathcal{N}_1 = \sum_{k=0}^{n_\mathrm{B}-1} \underbrace{\binom{n_\mathrm{B}}{k+1}}_{T_1} \underbrace{\left(\frac{t}{R}\right)^{k+1} \left(1 - \frac{t}{R}\right)^{n_\mathrm{B}-k-1}}_{T_2} \underbrace{\frac{k}{2}}_{T_3} \times \underbrace{(k+1)}_{T_4} \times \underbrace{\frac{1}{2}}_{T_5} = \frac{n_\mathrm{B}(n_\mathrm{B}-1)}{4} \left(\frac{t}{R}\right)^2. \tag{3}$$

Similarly, for $t > R$

$$\begin{aligned} \mathcal{N}_2 &= \sum_{k_1=0}^{n_\mathrm{B}-1} k_1 \mathcal{P}_\times(t \,|\, t > R) \\ &= \frac{n_\mathrm{B}(n_\mathrm{B}-1)}{2\pi R^2} \times \left(t^2 \arcsin\left(\frac{R}{t}\right) + 2R^2 \arccos\left(\frac{R}{t}\right) - R\sqrt{t^2 - R^2} \right). \end{aligned} \tag{4}$$

Intersection Density

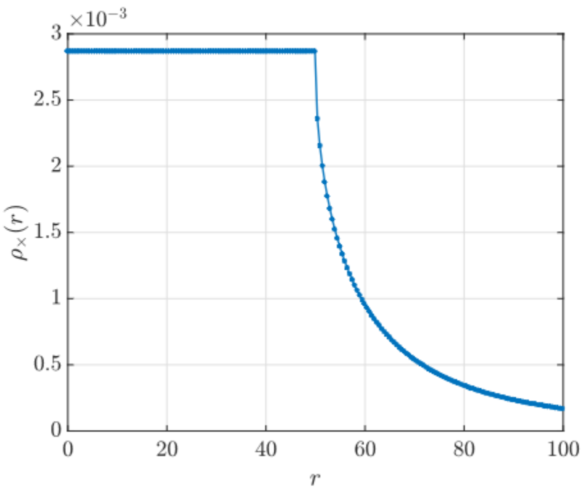

Figure: Intersection density for $R = 50$ and $n_\mathrm{B} = 10$.

Highlights of the Properties

- The line length density as well as the density of intersections remain constant at $\frac{n_\mathrm{B}}{2R}$ for $r \leq R$ and then decreases as $\mathcal{O}(1/r)$ as $r \to \infty$.

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- Other results skipped: distance to the nearest intersection.

Binomial Line Cox Process

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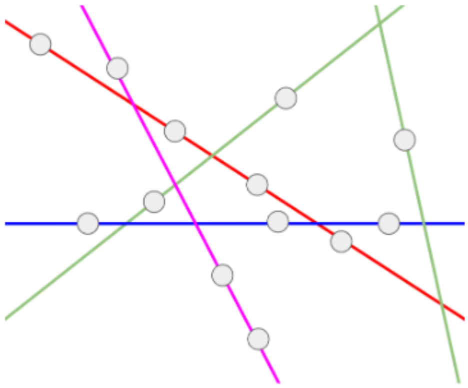

Nearest BLCP Point - Chord Length

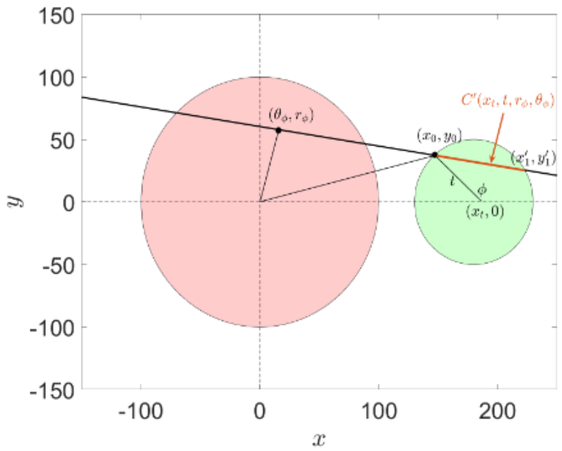

Figure: Illustration of the chord length $C'(x_t, t, r_\phi, \theta_\phi)$.

Void Probability

Theorem

The probability that the disk $\mathcal{B}((0, r_0), t)$ contains no points of Φ is given by

$$\mathcal{V}_{BLCP}\left(n_{\mathrm{B}}, \mathcal{B}((0, r_0), t)\right) = \left[\frac{1}{2\pi R} \iint_{\mathcal{D}(r_0,t)} \underbrace{\exp\left(-\lambda C(\theta, r)\right)}_{\mathcal{V}_{PPP}(\text{Chord Length})} \mathrm{d}r\, \mathrm{d}\theta\right]^{n_{\mathrm{B}}}, \quad (5)$$

where,

$$C(\theta, r) = \begin{cases} 2\sqrt{t^2 - (r_0\cos\theta - r)^2}; & t \geq |r_0 \cos\theta - r|, \\ 0; & \text{otherwise}, \end{cases} \quad (6)$$

is the length of the chord created by a line corresponding to $(\theta, r) \in \mathcal{D}$ in the disk $\mathcal{B}((0, r_0), t)$.

Void Probability

Corollary

[Distance Distribution] Following the void probability, the distance distribution of the nearest BLCP point from the test point $(0, r_0)$ *is*

$$F_{d_1}(t) = 1 - \mathcal{V}_{BLCP}\left(n_{\mathrm{B}}, \mathcal{B}((0, r_0), t)\right).$$

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From a wireless network perspective where the locations of the APs are modeled as points of a BLCP, the above result characterizes the distance distribution to the nearest AP.

Palm Perspective of the BLCP

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Lemma

For a BLCP Φ defined on a BLP $\mathcal{P}$ with n_{B} lines, we have

$$\mathbb{P}(\Phi \in Y \mid \mathbf{x} \in \Phi) = \mathbb{P}(\Phi_{n_{\mathrm{B}}-1} \cup \Phi_{\mathbf{x}} \cup \{\mathbf{x}\} \in \mathbf{Y}), \tag{7}$$

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Necessary to characterize the statistics of the performance metrics conditioned on an event that a node, e.g., a transmitter or an AP is located at a given point.

- Consider a network - APs $\sim$ BLCP.
- If a Rx at the origin associates with a Tx located at $\mathbf{x}$, then the interfering APs are located not only in the other $n_{\mathrm{B}} - 1$ lines but also on a line necessarily passing through $\mathbf{x}$.

Probability Generating Functional - Definitions

Definition

The **PGFL** of a point process Φ evaluated for a function ν is defined mathematically as the Laplace functional of $-\log \nu$ and is calculated as $\mathbb{E}\left[\prod_{\mathbf{x}\in\Phi} \nu(\mathbf{x})\right]$.

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$$\mathbb{E}\left[\prod_{\mathbf{x}\in\Phi_{PPP}}\nu(\mathbf{x})\right]=\exp\left(-\int_{\mathbb{R}^2}1-\nu(\mathbf{x})\Lambda(d\mathbf{x})\right) \tag{8}$$

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Let's look at an application with meta distributions.

Meta Distribution of the SINR with BLCP Nodes

Refresher On Indicators and Conditional Expectations

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- However, if Z is a function of other sources of randomness, then $\mathbb{E}[\mathbb{1}(Z > z)]$ does not reveal how the statistics of Z depend on those of the individual random elements.

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  - It has two parameters, the distribution has one parameter, and the average has zero parameters.

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Natural Progression and Back

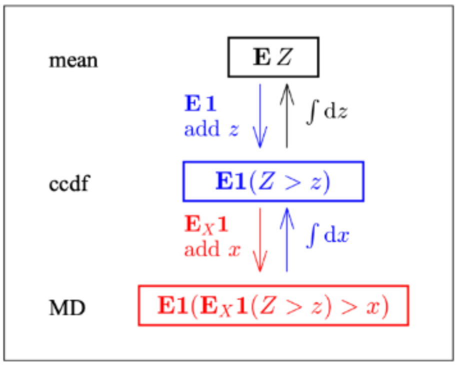

Figure: Going back and forth between mean, CCDF, and MD.

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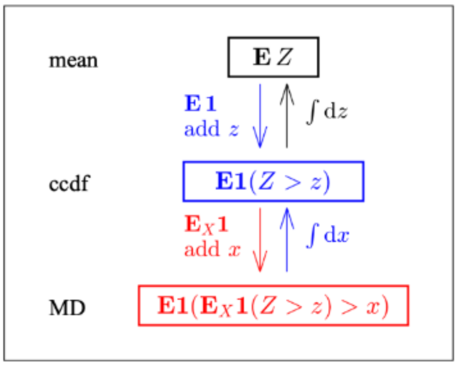

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Let $U = \mathbb{P}(Z > z|Y)$. Then, the mean of U is the distribution of Z and the distribution of U is the MD.

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Let us consider

$$Z = \frac{X}{Y}$$

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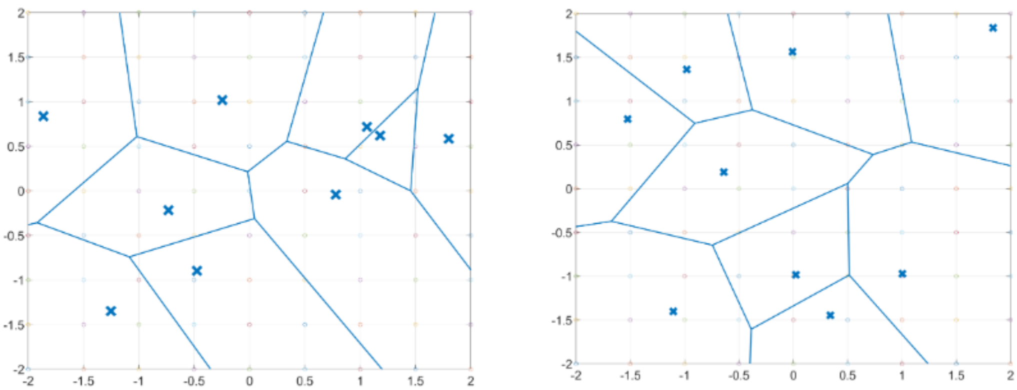
$$1 - x^{\frac{\mu_Y}{z\mu_X}} \tag{12}$$

Example in a Wireless Network



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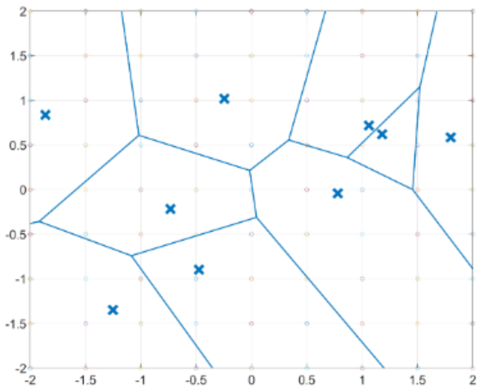

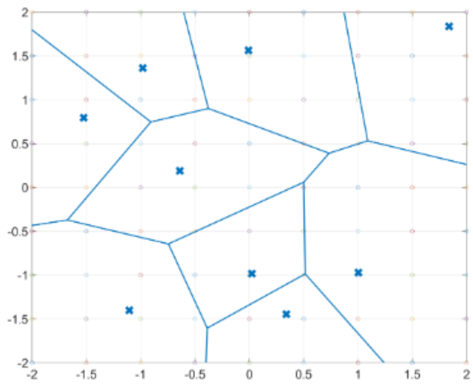

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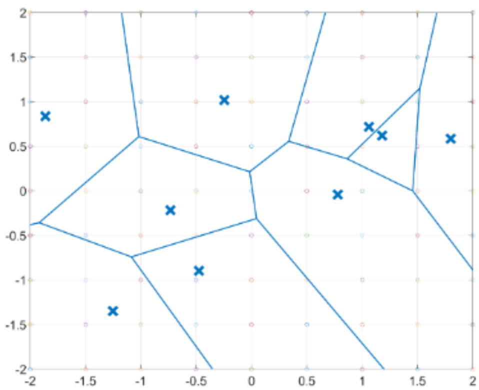

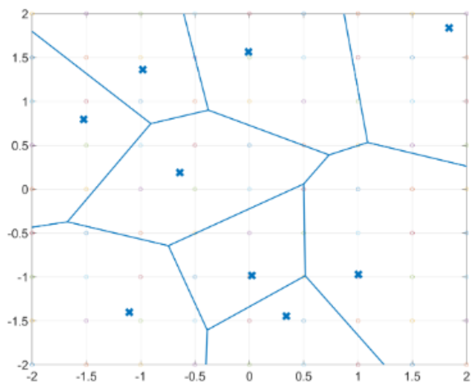

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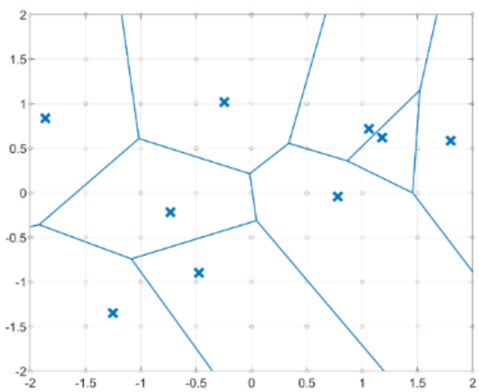

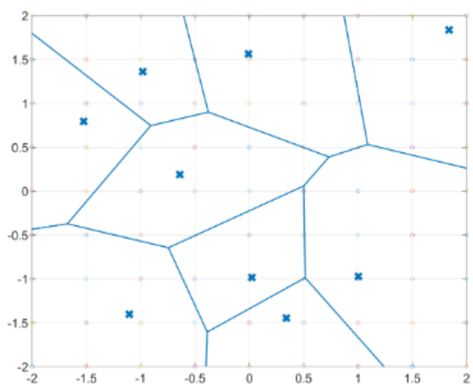

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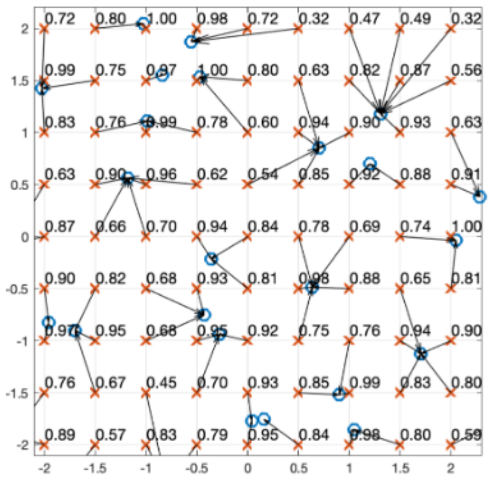

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- To know their distribution, we need to consult the MD.
- In contrast, without the MD, we have no information about the disparity between the users.
- Their personal probabilities could all be well concentrated around 0.76, or some could have probabilities near 0 and others near 1.
- Only the MD can reveal the performance of user percentiles, such as the "5% user" performance, which is the performance that 95% of the users achieve but 5% do not.

Let's return to BLCP - SINR Model

- Φ be a point process, the signal-to-interference-plus-noise ratio (SINR) $\xi(r_0)$ is

$$\xi(r_0) = \frac{\xi_0 ||\mathbf{x}_1||^{-\alpha} h_1}{1 + \xi_0 \sum_{\mathbf{x} \in \Phi \setminus \{\mathbf{x}_1\}} ||\mathbf{x}||^{-\alpha} h_\mathbf{x}}, \tag{13}$$

  ξ_0 is a constant that takes into account the transmit power, AWGN noise, path-loss constant, as well as the transmit and receive antenna gains.

For ease of notation, let us represent $||\mathbf{x}_i||$ by $d_\mathbf{x}$.

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- Φ be a point process, the SINR $\xi(r_0)$ is

$$\xi(r_0) = \frac{\xi_0 ||\mathbf{x_1}||^{-\alpha} h_1}{1 + \xi_0 \sum_{\mathbf{x} \in \Phi \setminus \{\mathbf{x_1}\}} ||\mathbf{x}||^{-\alpha} h_{\mathbf{x}}}, \quad (13)$$

  ξ_0 is a constant that takes into account the transmit power, AWGN noise, path-loss constant, as well as the transmit and receive antenna gains.

For ease of notation, let us represent $||\mathbf{x_i}||$ by $d_{\mathbf{x}}$.
Furthermore, let us assume ALOHA (p).

Conditional Success Probability

The conditional success probability, i.e.,

$$P_s(\gamma) = \mathbb{P}(\xi(r_0) \geq \gamma | \Phi)$$

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Its CCDF, called the meta distribution of the SINR, is given as

$$\mathcal{P}_{\mathrm{M}}(\gamma, \beta) = \mathbb{P}\left(P_s(\gamma) \geq \beta\right) = \mathbb{P}\left(\mathbb{P}(\xi(r_0) \geq \gamma | \Phi) \geq \beta\right). \quad (14)$$

which is a function of two parameters $\gamma \geq 0$ and $0 \leq \beta \leq 1$.

Conditional Success Probability

Let the set of locations of the interfering nodes be denoted by $\mathcal{C} \subset \Phi'$.

$$P_s(\gamma) = \mathbb{P}\left(\xi(r_0) \geq \gamma \mid \Phi\right) = \mathbb{P}\left[\frac{\xi_0 d_1{}^{-\alpha} h_1}{1 + \xi_0 \sum_{\mathbf{x} \in \Phi'} h_{\mathbf{x}} d_{\mathbf{x}}^{-\alpha} \mathbf{1}(\mathbf{x} \in \mathcal{C})} \geq \gamma \mid \Phi\right]$$

$$= \mathbb{P}\left[h_1 > \frac{\gamma + \gamma\xi_0 \sum_{\mathbf{x} \in \Phi'} h_{\mathbf{x}} d_{\mathbf{x}}^{-\alpha} \mathbf{1}(\mathbf{x} \in \mathcal{C})}{\xi_0 d_1^{-\alpha}}\right] \stackrel{(a)}{=} \mathbb{E}_{h_{\mathbf{x}}}\left[\exp\left(\frac{-\gamma - \gamma\xi_0 \sum_{\mathbf{x} \in \Phi'} h_{\mathbf{x}} d_{\mathbf{x}}^{-\alpha} \mathbf{1}(\mathbf{x} \in \mathcal{C})}{\xi_0 d_1^{-\alpha}}\right)\right]$$

$$= \exp\left(\frac{-\gamma}{\xi_0 d_1^{-\alpha}}\right) \mathbb{E}_{h_{\mathbf{x}}}\left[\exp\left(\frac{-\gamma\xi_0 \sum_{\mathbf{x} \in \Phi'} h_{\mathbf{x}} d_{\mathbf{x}}^{-\alpha} \mathbf{1}(\mathbf{x} \in \mathcal{C})}{\xi_0 d_1^{-\alpha}}\right)\right]$$

$$= \exp\left(\frac{-\gamma}{\xi_0 d_1^{-\alpha}}\right)\left(\prod_{\mathbf{x} \in \Phi'} p \mathbb{E}_{h_{\mathbf{x}}} \exp\left(\frac{-\gamma\xi_0 d_{\mathbf{x}}^{-\alpha} h_z}{\xi_0 d_1^{-\alpha}}\right) + 1 - p\right)$$

$$\stackrel{(b)}{=} \exp\left(\frac{-\gamma}{\xi_0 d_1^{-\alpha}}\right)\left(\prod_{\mathbf{x} \in \Phi'} \frac{p}{1 + \frac{\gamma d_{\mathbf{x}}^{-\alpha}}{d_1^{-\alpha}}} + 1 - p\right).$$

Moments

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- The standard approach to circumvent this challenge is by first deriving its moments and then transforming them to the distribution.
- Moments reveal key features:
  - First moment: $M_1 = \mathbb{E}_\Phi\left[P_s(\gamma)\right] \rightarrow$ standard success probability.
  - -1^{th} moment: $M_{-1} = \mathbb{E}_\Phi\left[\frac{1}{P_s(\gamma)}\right] \rightarrow$ mean local delay.
  - Similarly, variance etc.

First Moment in a BLCP

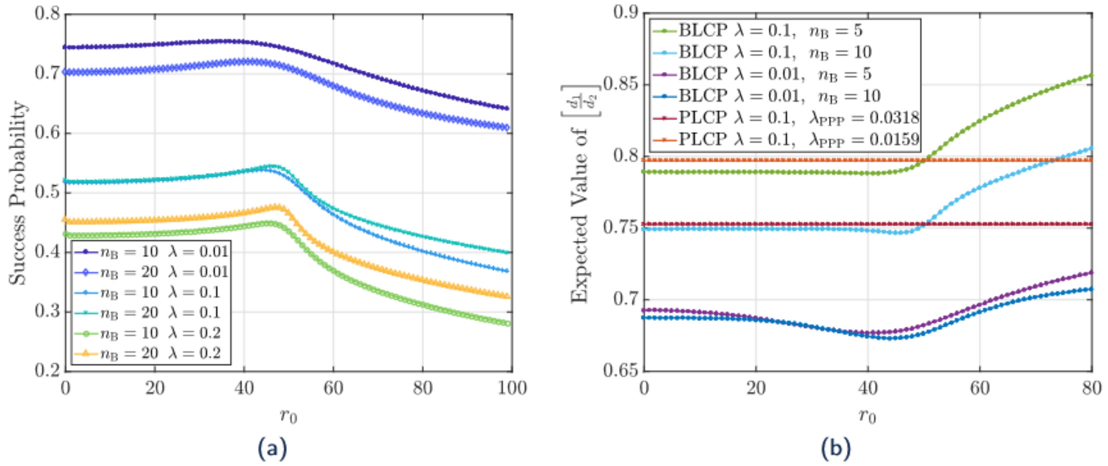


Figure: (a) Success probability with respect to r_0. Here $R = 50$. (b) $\mathbb{E}\left[\frac{d_1}{d_2}\right]$ with respect to r_0.

-1^{th} Moment

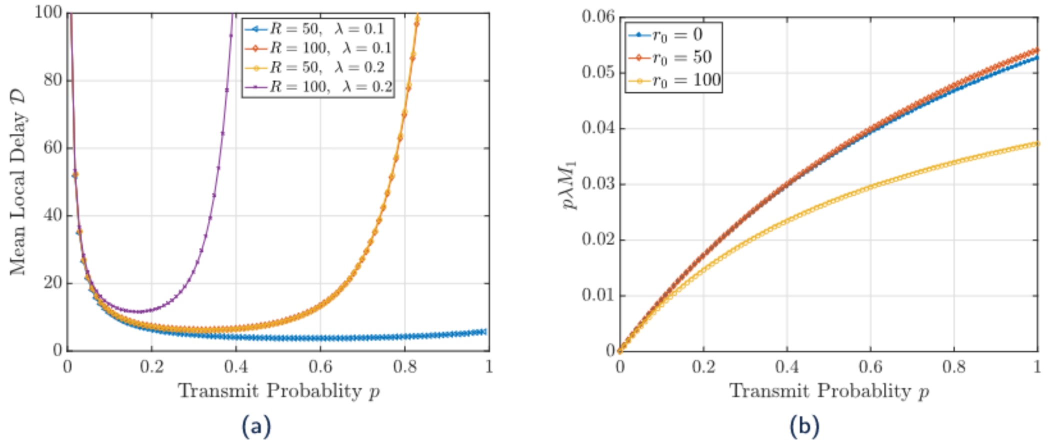
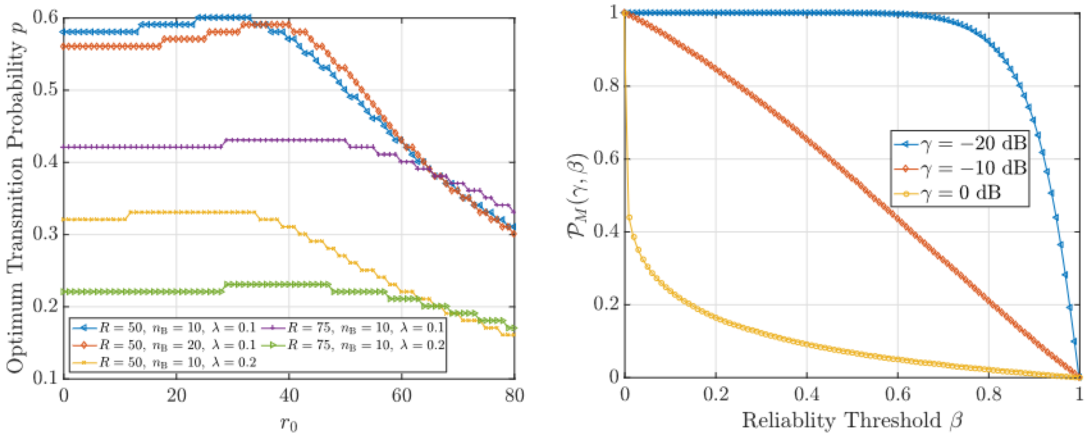


Figure: (a) Mean local delay with respect to the transmit probability for different values of R and λ. Here, $r_0 = 0$. (b) Successful transmission density.

Optimal Access Probability



(a) (b)

Figure: (a) Optimal transmit probability for minimizing the mean local delay. (b) SINR meta distribution.

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  - Rigorously studied in Part 2.
- Questions:
  - How accurate is BLP/BLCP? - Working on it.
  - Nearest point in the L_1 sense? Percolation questions.
  - How to integrate the BLP model with existing street models?

Thanks!

Questions?
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